Avoiding Communication in Dense Linear Algebra

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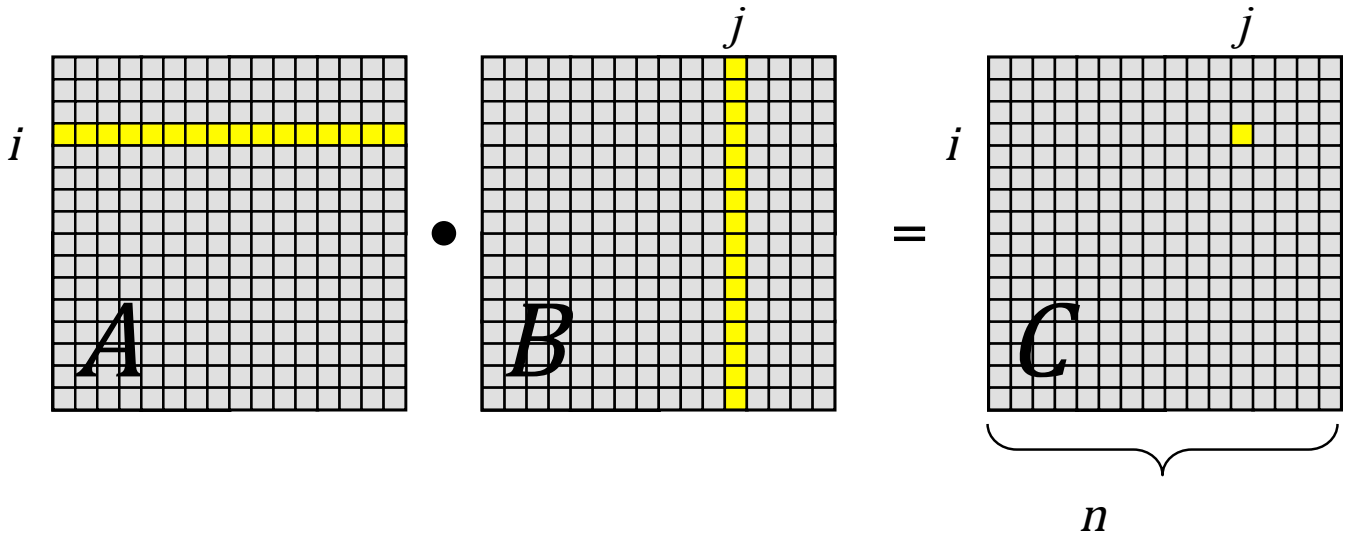
Let's start with matrix multiplication

Suppose we want to compute

$$A \cdot B = C$$

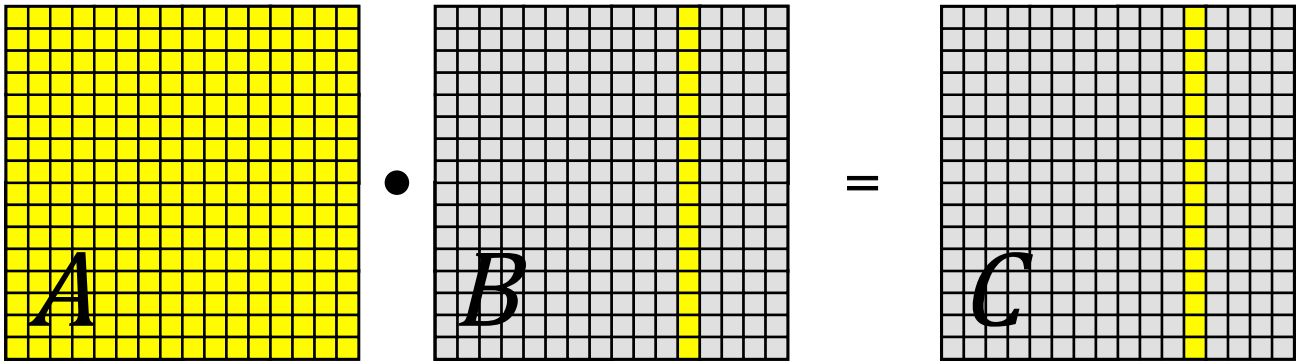
where A and B are $n \times n$ matrices



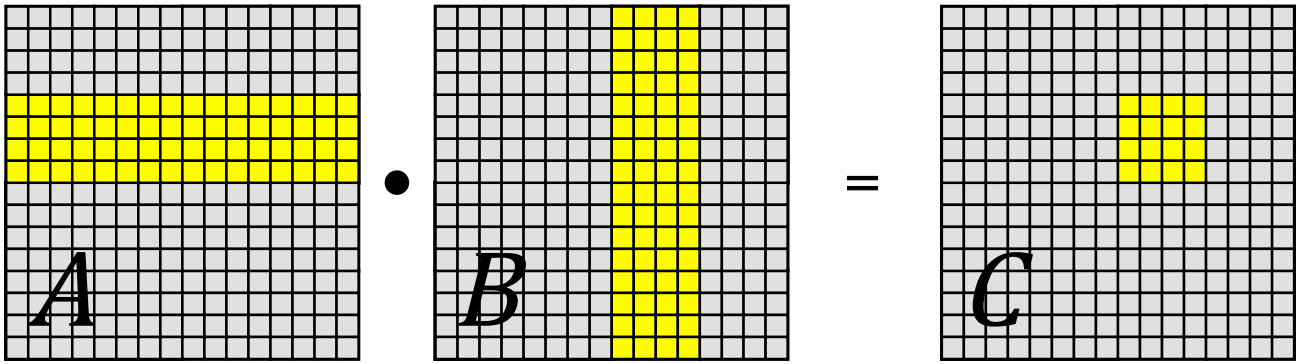
$$\sum_{k=1}^{n} a_{ik} b_{kj} = c_{ij}$$

Two algorithms for matrix multiplication...

We can multiply matrices like this ("matrix-vector" algorithm):

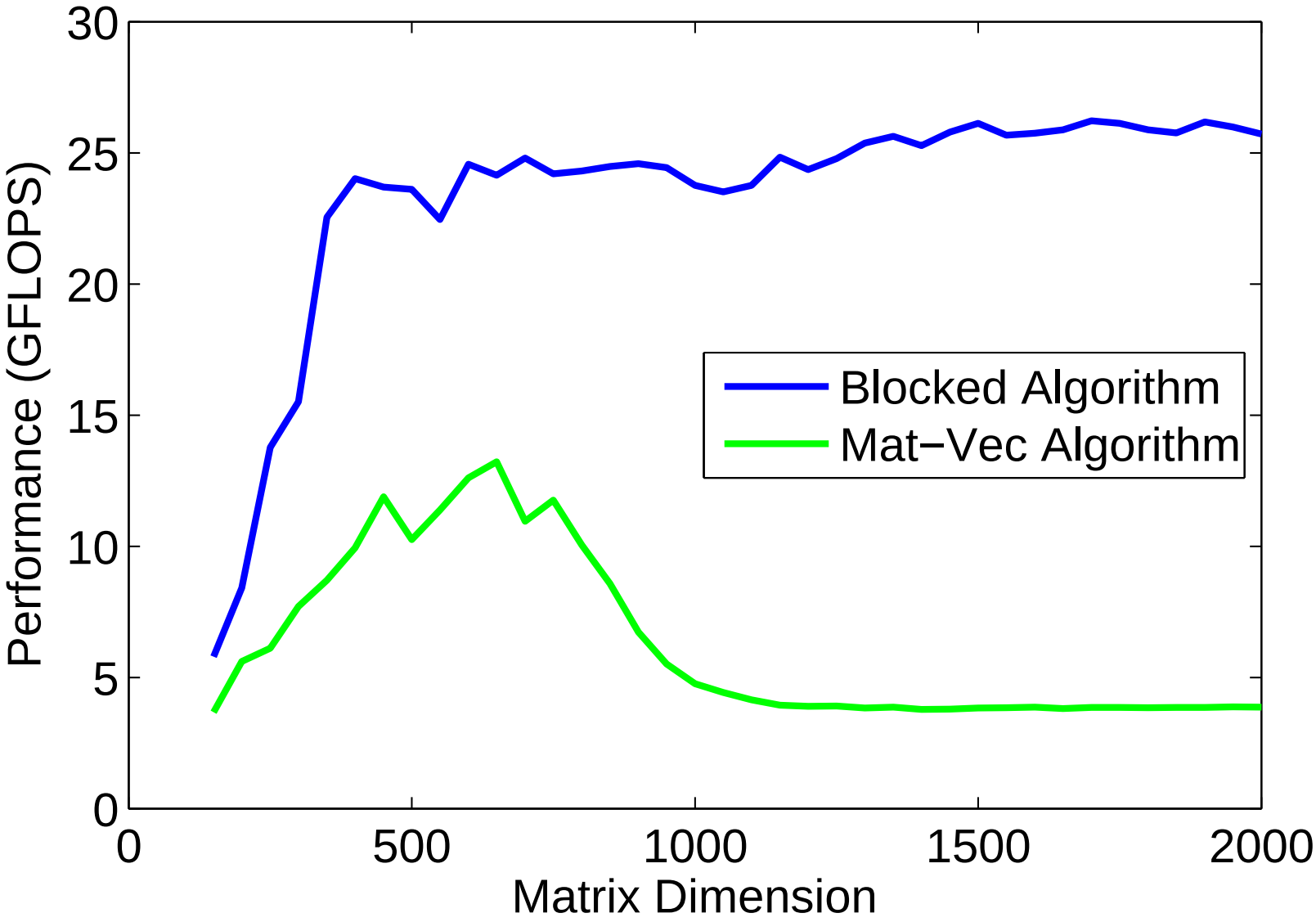


or like this ("blocked" algorithm):



In both cases, we do $2n^3 + O(n^2)$ flops

Same computation, different performance

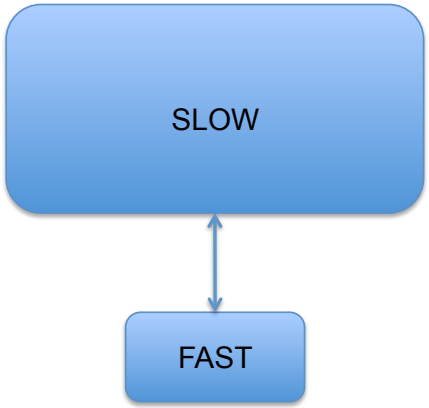


We must consider communication

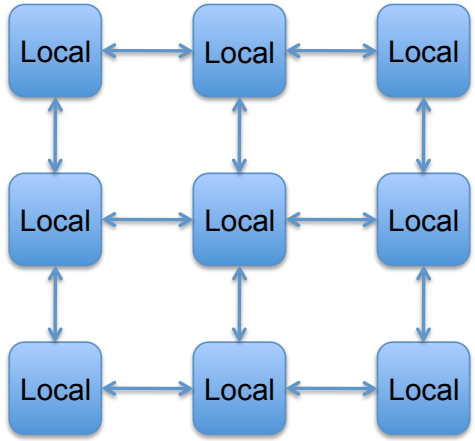
By *communication*, I mean

- moving data within memory hierarchy on a sequential computer
- moving data between processors on a parallel computer

For high-level analysis, we'll use these simple memory models:



Sequential



Parallel

Runtime model

Measure computation in terms of *# flops* performed

Time per flop: γ

Measure communication in terms of *# words* communicated

Time per word: β

Total running time of an algorithm (ignoring overlap):

$$\gamma \cdot (\text{\# flops}) + \beta \cdot (\text{\# words})$$

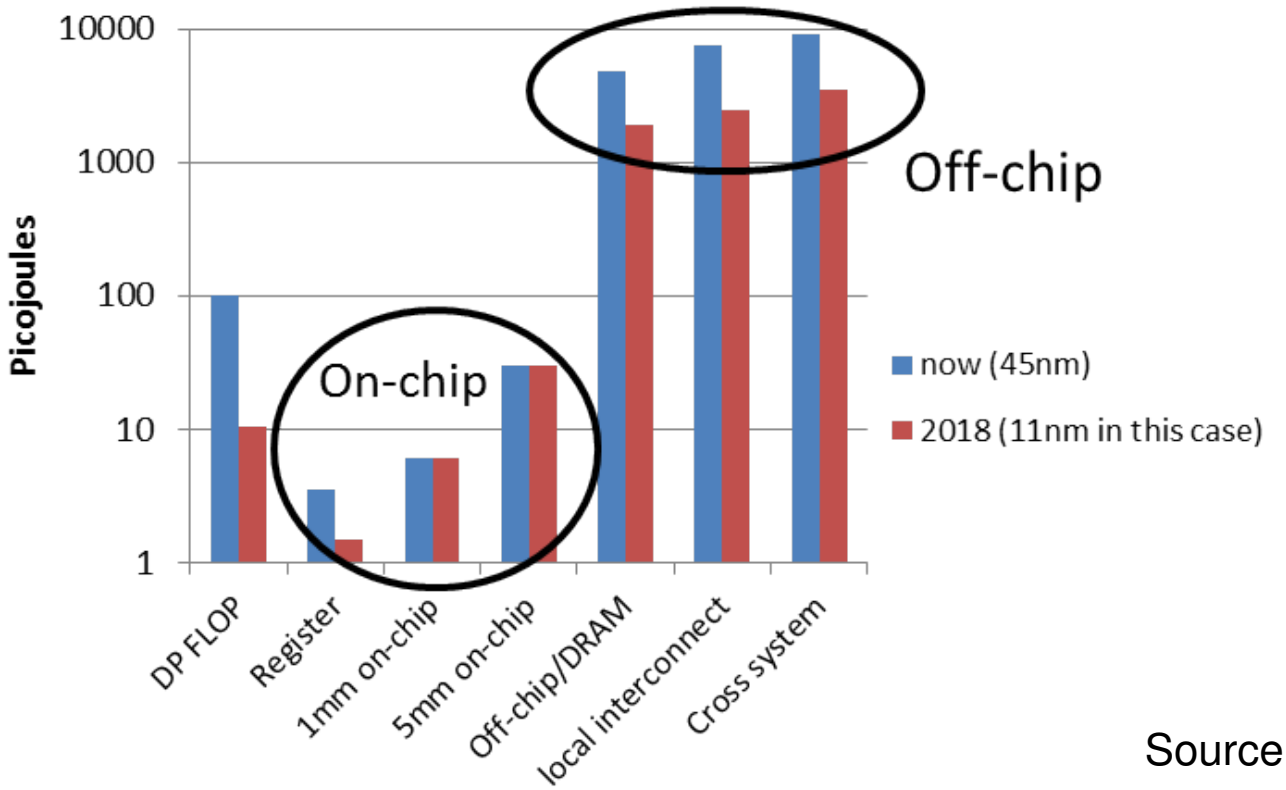
$\beta \gg \gamma$ as measured in time *and* energy, and the relative cost of communication is increasing

Why avoid communication

Annual Improvements in Time

Flop rate γ	DRAM Bandwidth β	Network Bandwidth β
59% per year	23% per year	26% per year

Energy cost comparisons

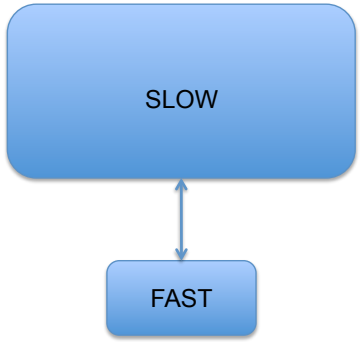


Source: John Shalf

Costs of matrix multiplication algorithms

Let M be the size of the fast memory

The blocked algorithm uses a block size of $\sqrt{M/3}$



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Mat-Vec Algorithm	$O(n^3)$	$O(n^3)$
Blocked Algorithm	$O(n^3)$	$O\left(\frac{n^3}{\sqrt{M}}\right)$

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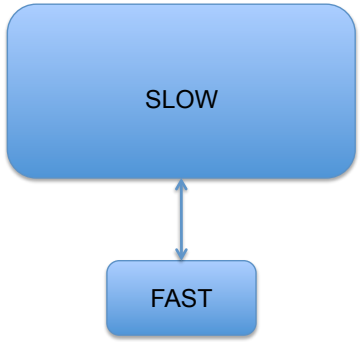

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Can we do better than the blocked algorithm?

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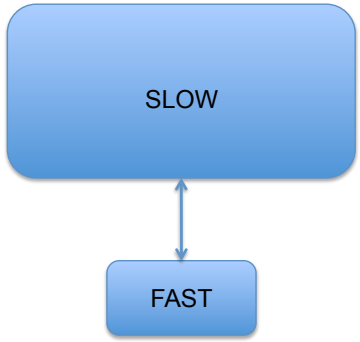

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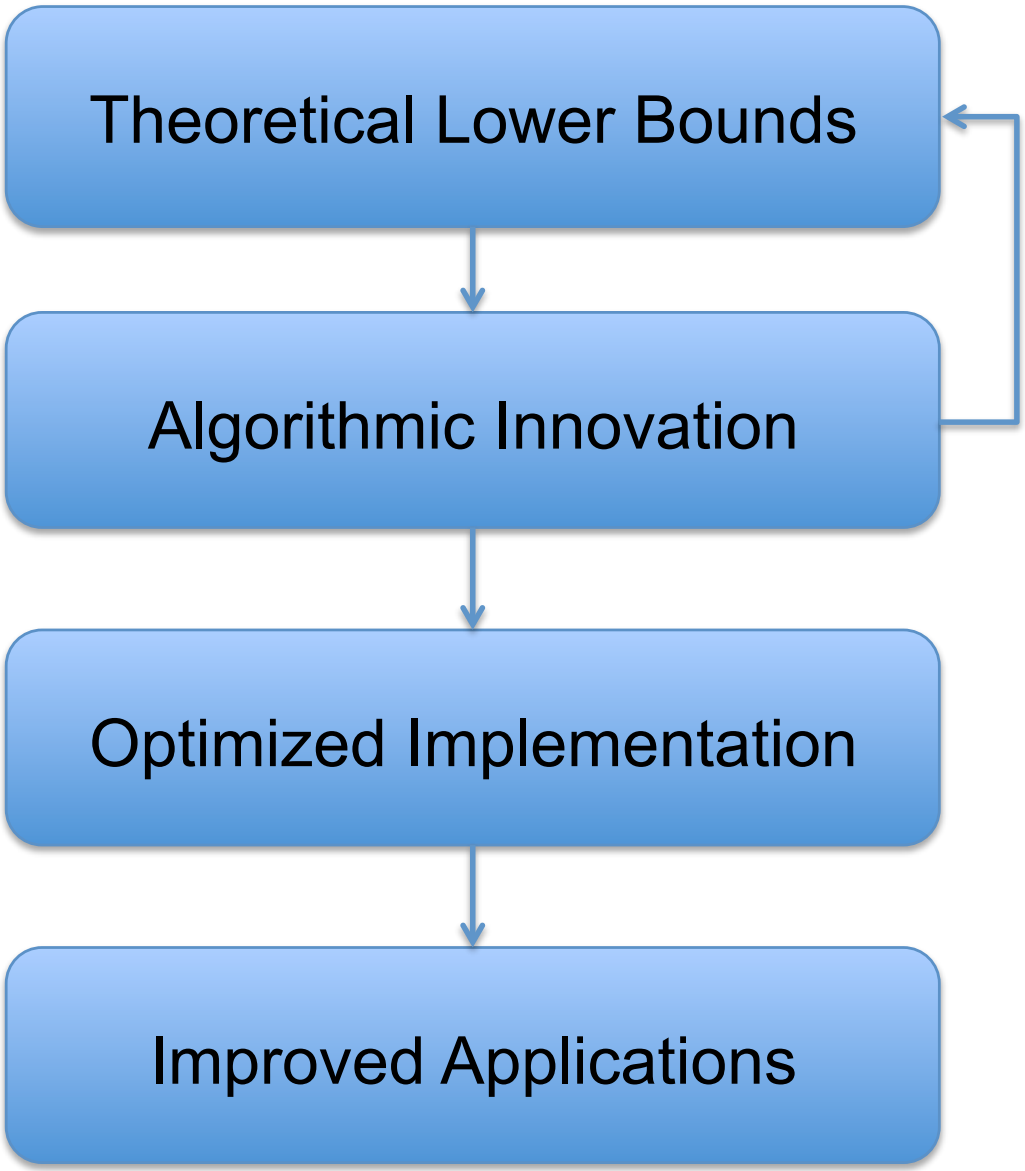
Can we do better than the blocked algorithm?

No . . . and Yes

Summary

- some communication is necessary: we can prove **lower bounds**
  - for Strassen's matrix multiplication*
  - for "classical" dense linear algebra
- theoretical analysis identifies sub-optimal algorithms and spurs **algorithmic innovation**
  - parallel implementation of Strassen's matrix multiplication*
  - solving symmetric indefinite linear system*
  - computing eigenvalues of a symmetric band matrix
  - computing a tall-skinny SVD
  - LU and QR factorizations
  - nonsymmetric eigendecompositions
- minimizing communication leads to speedups in practice

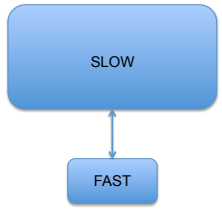
Theory to practice



Lower bounds for classical matrix multiplication

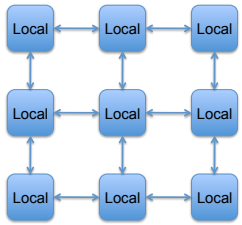
- Assume $O(n^3)$ algorithm
- Sequential case with fast memory of size M
  - lower bound on words moved between fast/slow mem:

$$\Omega\left(\frac{n^3}{\sqrt{M}}\right) \quad \text{[Hong \& Kung 81]}$$



  - attained by blocked algorithm
- Parallel case with P processors (local memory of size M)
  - lower bound on words communicated (along critical path):

$$\Omega\left(\frac{n^3}{P\sqrt{M}}\right) \quad \text{[Toledo et al. 04]}$$



  - also attainable

Let's ask again:

Can we do better than the blocked algorithm?

Given the computation involved, it minimized communication. . .

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Can we do better than the blocked algorithm?

Given the computation involved, it minimized communication. . .

. . . but what if we change the computation?

It's possible to reduce both computation *and* communication

Strassen's algorithm for matrix multiplication

Strassen showed how to use 7 multiplies instead of 8 for 2×2 multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Classical Algorithm

$$\begin{aligned}
M_1 &= A_{11} \cdot B_{11} \\
M_2 &= A_{12} \cdot B_{21} \\
M_3 &= A_{11} \cdot B_{12} \\
M_4 &= A_{12} \cdot B_{22} \\
M_5 &= A_{21} \cdot B_{11} \\
M_6 &= A_{22} \cdot B_{21} \\
M_7 &= A_{21} \cdot B_{12} \\
M_8 &= A_{22} \cdot B_{22} \\
C_{11} &= M_1 + M_2 \\
C_{12} &= M_3 + M_4 \\
C_{21} &= M_5 + M_6 \\
C_{22} &= M_7 + M_8
\end{aligned}$$

Strassen's Algorithm

$$\begin{aligned}
M_1 &= (A_{11} + A_{22}) \cdot (B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22}) \cdot B_{11} \\
M_3 &= A_{11} \cdot (B_{12} - B_{22}) \\
M_4 &= A_{22} \cdot (B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12}) \cdot B_{22} \\
M_6 &= (A_{21} - A_{11}) \cdot (B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) \\
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
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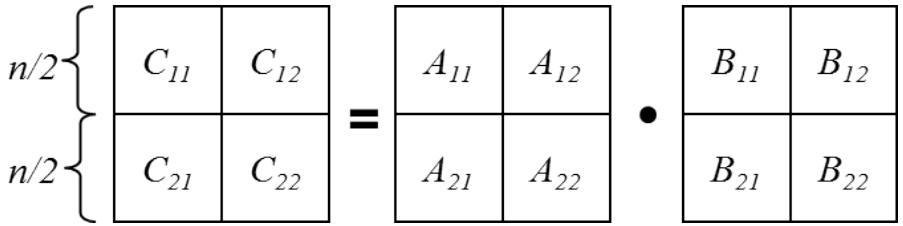

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\end{aligned}
$$

Flop count recurrence:

$$F(n) = 7 \cdot F(n/2) + \Theta(n^2)$$

$$F(n) = \Theta\left(n^{\log_2 7}\right)$$

$$\log_2 7 \approx 2.81$$

$$
\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
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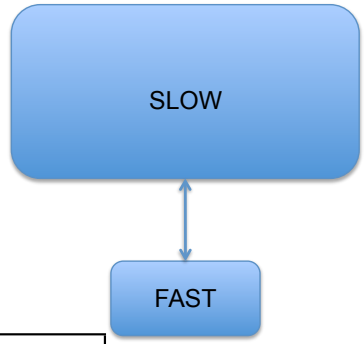
Sequential communication costs

If you implement Strassen's algorithm recursively on a sequential computer:



	Computation # flops	**Communication # words**
Classical (blocked)	$O(n^3)$	$O\left(\left(\frac{n}{\sqrt{M}}\right)^3 M\right)$
Strassen	$O(n^{\log_2 7})$	$O\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} M\right)$

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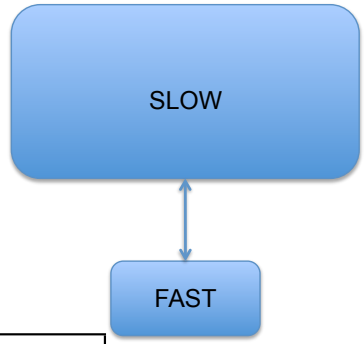

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Can we reduce Strassen's communication cost further?

Lower bounds for Strassen's algorithm

Theorem (Ballard, Demmel, Holtz, Schwartz 12)

On a sequential machine, Strassen's algorithm must communicate

$$\#\ words = \Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} M\right)$$

and on a parallel machine, it must communicate

$$\#\ words = \Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} \frac{M}{P}\right)$$

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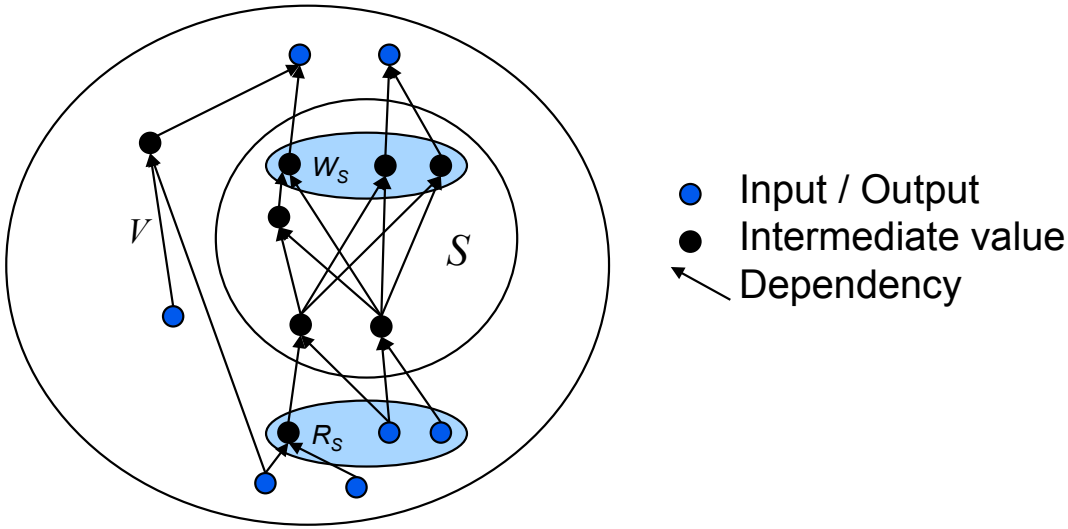
This work

- received the *SPAA Best Paper Award* [BDHS11]
- appeared in the *Journal of the ACM* [BDHS12a]
- and has been invited to appear as a Research Highlight in the *Communications of the ACM*

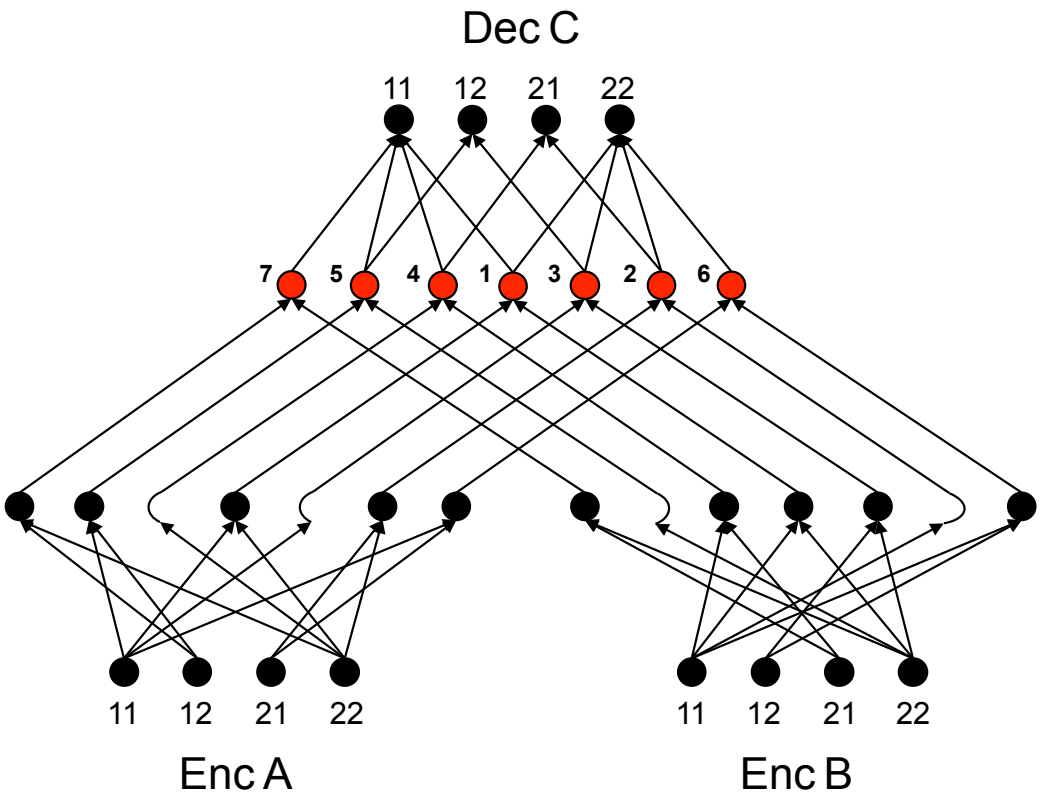
Computation graph analysis



We connected graph *expansion* to communication

- expansion describes the relationship between a subset and its neighbors in the complement
- larger expansion implies more communication necessary

Strassen's computation graph



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Optimal Parallel Algorithm?

This lower bound proves that the sequential recursive algorithm is communication-optimal

What about the parallel case?

Optimal Parallel Algorithm?

This lower bound proves that the sequential recursive algorithm is communication-optimal

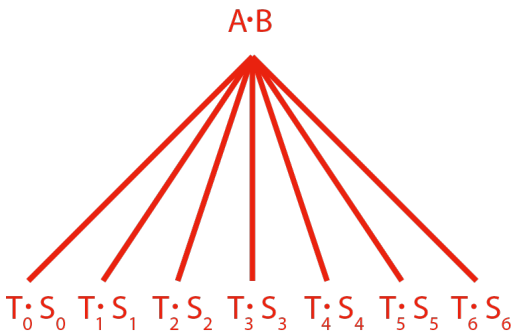
What about the parallel case?

- Earlier attempts to parallelize Strassen had communication costs which exceeded the lower bound
- We developed a new algorithm that is communication-optimal, called Communication-Avoiding Parallel Strassen (CAPS) [BDH+12b]

Main idea of CAPS algorithm

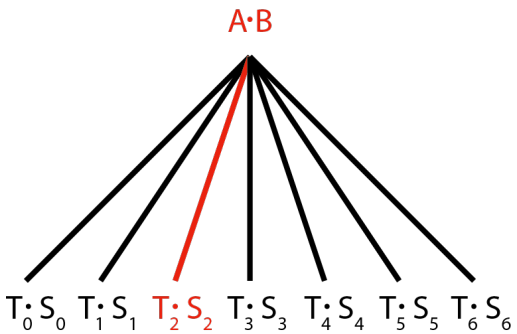
At each level of recursion tree, choose either breadth-first or depth-first traversal of the recursion tree

Breadth-First-Search (BFS)



- Runs all 7 multiplies in parallel
  - each uses $P/7$ processors
- Requires $7/4$ as much extra memory
- Requires communication, but minimizes communication in subtrees

Depth-First-Search (DFS)



- Runs all 7 multiplies sequentially
  - each uses all P processors
- Requires $1/4$ as much extra memory
- Increases communication by factor of $7/4$ in subtrees

Is it optimal?

After algorithmic analysis, we can compare communication costs to the lower bound:

	Communication # words
CAPS	$O\left(\max\left\{\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7}\frac{M}{P}, \frac{n^2}{P^{2/\log_2 7}}\right\}\right)$
Lower Bound [BDHS11]	$\Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7}\frac{M}{P}\right)$

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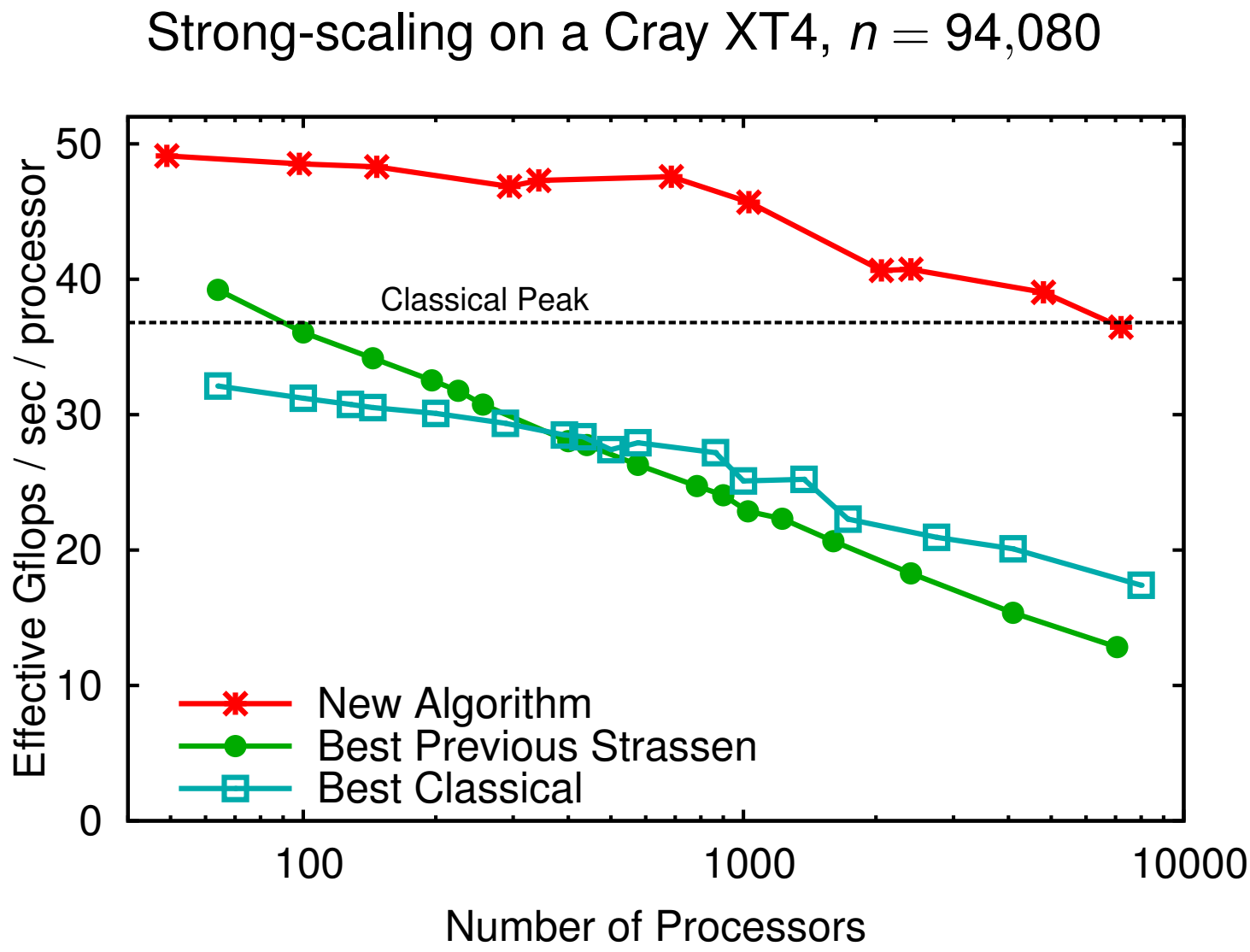
New Lower Bound [BDH+12a]	$\Omega\left(\frac{n^2}{P^{2/\log_2 7}}\right)$

Performance of CAPS on a large problem



- Actual Peak
- Strassen-Winograd peak
- Performance Model

Can we beat Strassen?

Strassen's algorithm allows for less computation and communication than the classical $O(n^3)$ algorithm

We have algorithms that attain its communication lower bounds and perform well on highly parallel machines

Can we do any better?

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Can we do any better?

Yes, but there are other complications

Let's go back to classical matrix algorithms

Lower bounds for classical matrix multiplication

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- Sequential case with fast memory of size M
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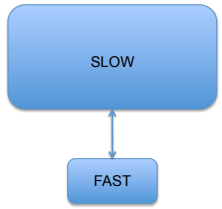

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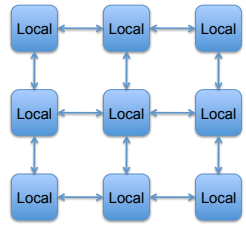

  - also attainable

Extensions to the rest of linear algebra

Theorem (Ballard, Demmel, Holtz, Schwartz 11)

If a computation "smells" like 3 nested loops, it must communicate

$$\#\ words = \Omega\left(\frac{\#\ flops}{\sqrt{memory\ size}}\right)$$

This result applies to

- dense or sparse problems
- sequential or parallel computers

This work was recognized with the *SIAM Linear Algebra Prize*, given to the best paper from the years 2009-2011

▸ Main Idea

Extensions to the rest of linear algebra

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What smells like 3 nested loops?

- the rest of BLAS 3 (e.g. matrix multiplication, triangular solve)
- Cholesky, LU, LDL^T, LTL^T decompositions
- QR decomposition
- eigenvalue and SVD reductions
- sequences of algorithms (e.g. repeated matrix squaring)
- graph algorithms (e.g. all pairs shortest paths)

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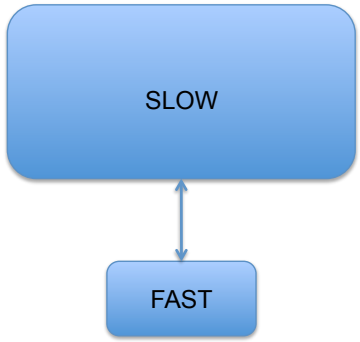
What if the computation smells like 5 nested loops?

. . . come see Nick's talk next week

▸ Main Idea

Optimal algorithms - sequential $O(n^3)$ linear algebra



Computation	Optimal Algorithm
BLAS 3	blocked algorithms [Gustavson 97]
Cholesky	LAPACK [Ahmed & Pingali 00] [BDHS10]
Symmetric Indefinite	LAPACK (rarely) [BDD+12a]
LU	LAPACK (rarely) [Toledo 97]* [Grigori et al. 11]
QR	LAPACK (rarely) [Frens & Wise 03] [Elmroth & Gustavson 98]* [Hoemmen et al. 12]*
Eig, SVD	[BDK12a], [BDD12b]

▸ Parallel Case

Example: symmetric indefinite linear solve

Suppose we want to solve $Ax = b$ where A

- is symmetric (save half the storage and flops)
- but indefinite (need to permute rows/cols for numerical stability)

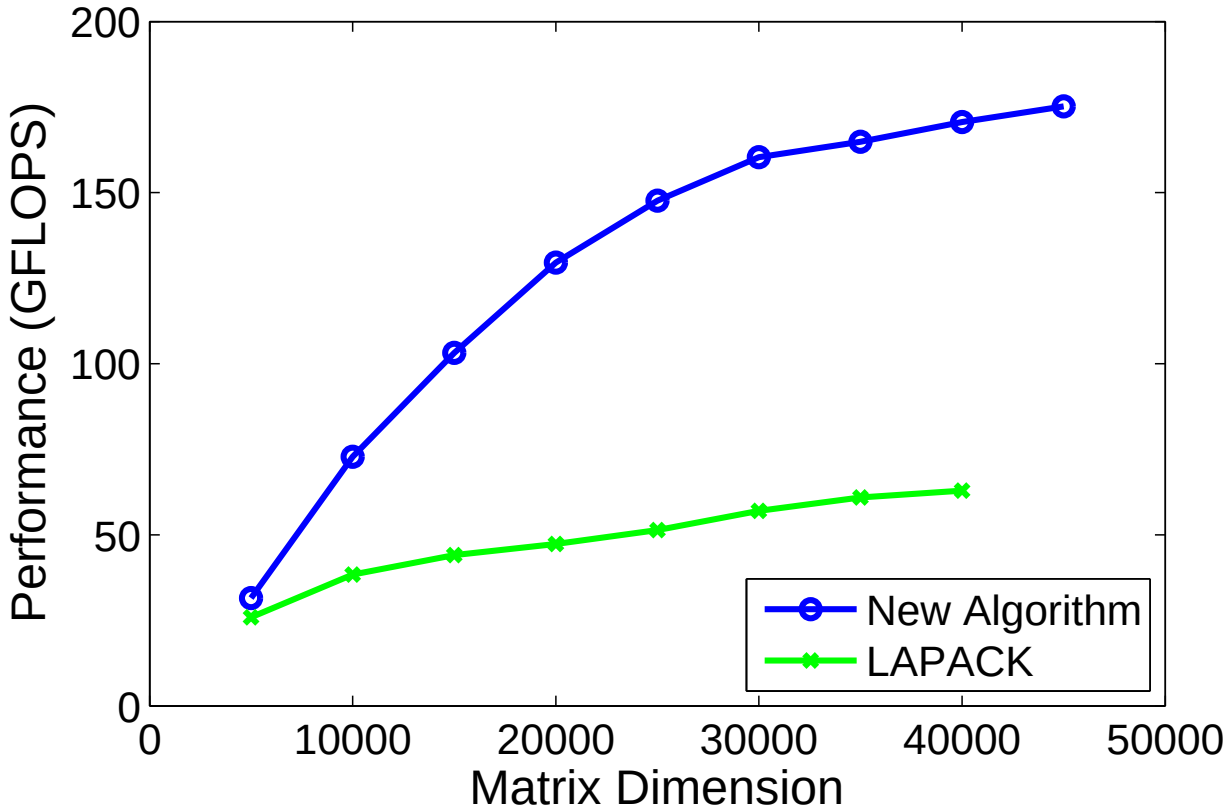
We generally want to compute a factorization

$$PAP^T = LTL^T$$

P is a permutation, L is triangular, and T is symmetric and "simpler"

Reducing communication improves performance

Performance of symmetric indefinite linear system solvers



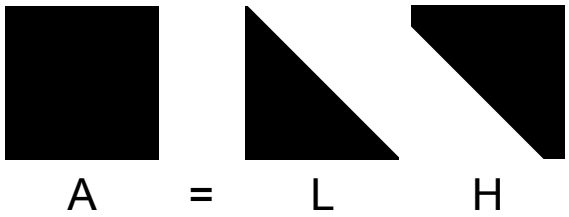
Implemented within PLASMA library [BBD+13]

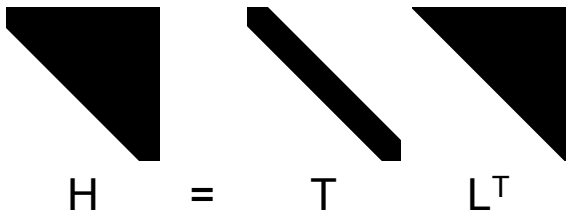
This work will receive a *Best Paper Award* at IPDPS '13

Aasen's symmetric indefinite factorization

We're solving $Ax = b$ where $A = A^T$ but A is indefinite

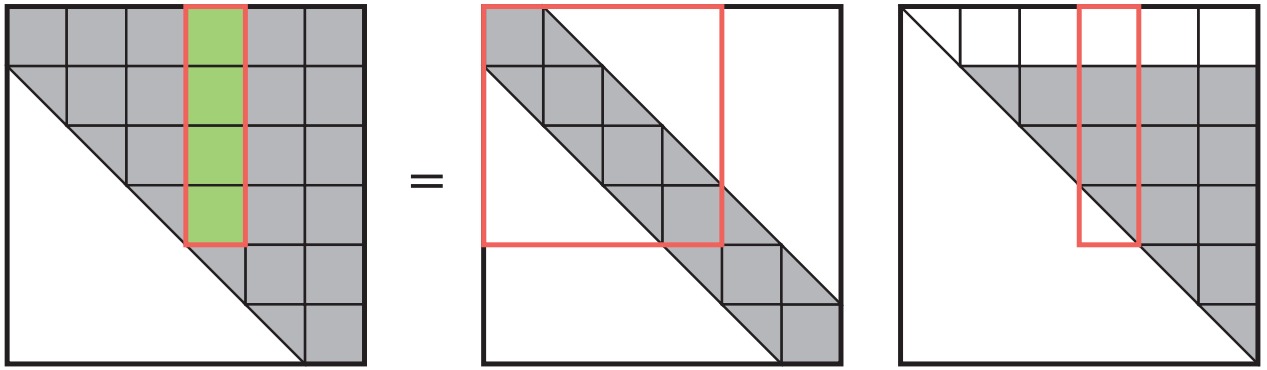
- Standard approach is to compute $PAP^T = LDL^T$
  - L is lower triangular and D is block diagonal (1×1 and 2×2 blocks)
  - requires complicated pivoting, harder to do tournament pivoting
- Aasen's approach is to compute $PAP^T = LTL^T$ [Aas71]
  - L is lower triangular and T is tridiagonal
  - pivoting is more like LU (nonsymmetric case)



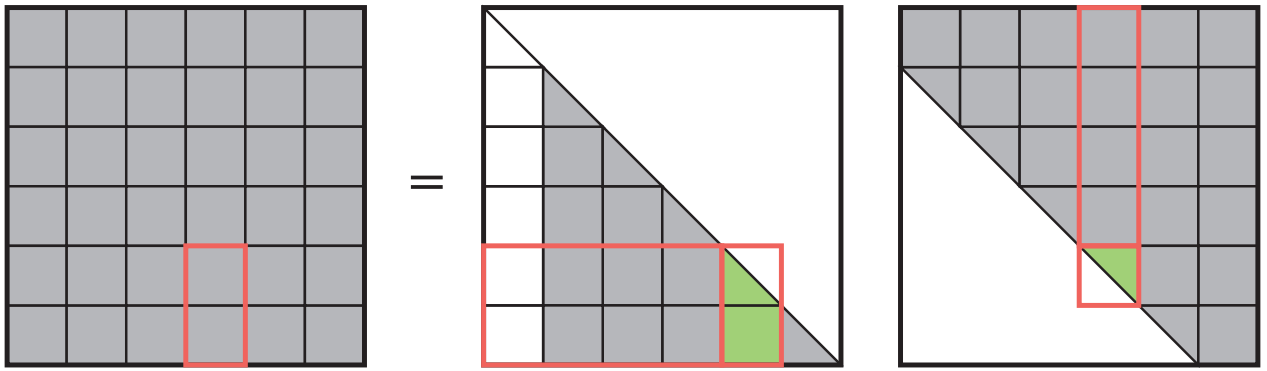


Blocked version of Aasen's algorithm

Compute block column of H from T and L:



Compute block column of L and subdiagonal block of H with LU:

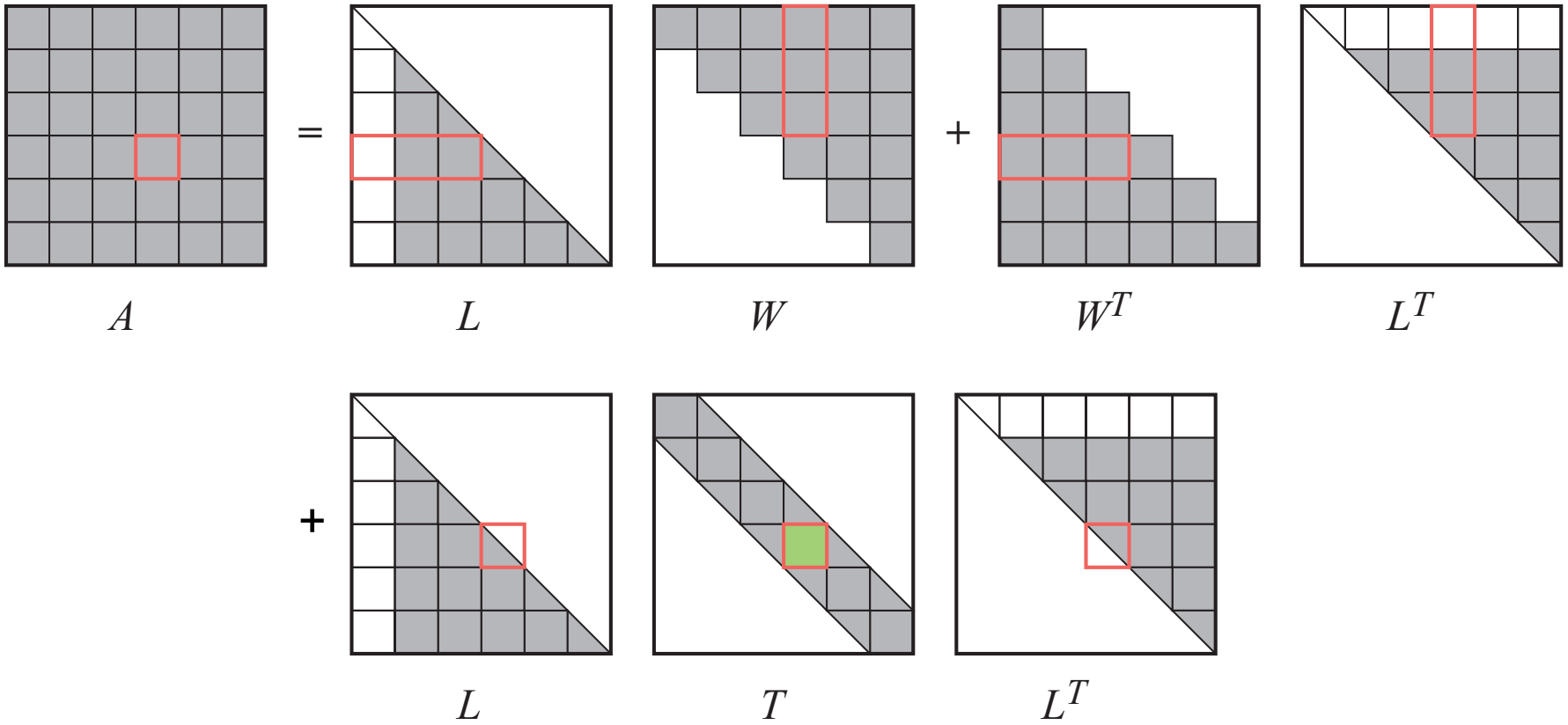


Converting scalar to blocked algorithm...

1. $\underline{H_{1:J-1,J}} = T_{1:J-1,1:J-1} L^T_{1:J-1,J}$
2. $A_{J,J} = L_{J,1:J-1} H_{1:J-1,J} + L_{J,J} \underline{H_{J,J}}$
3. $H_{J,J} = T_{J,J-1} L^T_{J-1,J} + \underline{T_{J,J}} L^T_{J,J}$
4. $A_{J+1:N,J} = L_{J+1:N,1:J} H_{1:J,J} + \underline{L_{J+1:N,J+1} H_{J+1,J}}$
5. $H_{J+1,J} = \underline{T_{J+1,J}} L^T_{J,J}$

Computing symmetric blocks of T

Since diagonal blocks of T are symmetric, need to be computed from a symmetric equation, which includes two-sided triangular solve:



$W \sim H$

Other complications

- How to do tall-skinny LU decomposition?
  - use tournament pivoting
  - use recursive algorithm
  - use LAPACK algorithm
- Need to take care in applying symmetric permutations
- We still need to decompose band matrix T:
  - non-symmetric band LU decomposition
  - successive band reduction (orthogonal similarity transformations)
  - Kaufman's symmetric retraction algorithm

Comm-optimal symmetric indefinite factorization

- After handling all the complications, we obtain a blocked version of Aasen's algorithm which moves

$$O\left(\frac{n^3}{\sqrt{M}}\right) \text{ words}$$

  and matches the communication lower bound

- A shared-memory implementation in the PLASMA library outperforms the best implementation of the standard algorithm

Summary

- some communication is necessary: we can prove **lower bounds**
  - for Strassen's matrix multiplication*
  - for "classical" dense linear algebra
- theoretical analysis identifies sub-optimal algorithms and spurs **algorithmic innovation**
  - parallel implementation of Strassen's matrix multiplication*
  - solving symmetric indefinite linear system*
  - computing eigenvalues of a symmetric band matrix
  - computing a tall-skinny SVD
  - LU and QR factorizations
  - nonsymmetric eigendecompositions
- minimizing communication leads to speedups in practice

Collaborators

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- Ioana Dumitriu (U Washington)
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- Tamara Kolda (Sandia NL)
- Benjamin Lipshitz (UC Berkeley)
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- Oded Schwartz (UC Berkeley)
- Edgar Solomonik (UC Berkeley)
- Sivan Toledo (Tel-Aviv U)
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Avoiding Communication in Dense Linear Algebra

Grey Ballard

Thank You!

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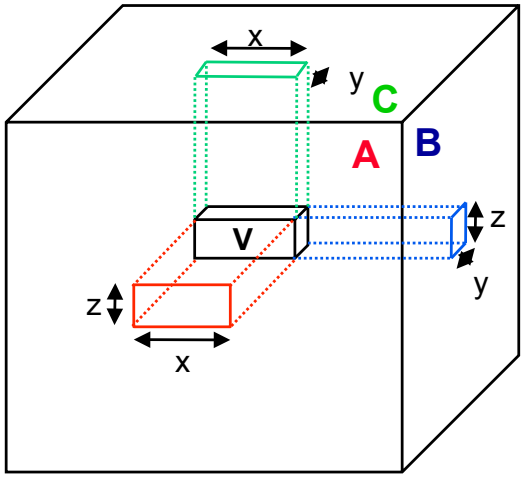
Other Ongoing and Future Projects

- implementing these algorithms in communication-bound settings
  - e.g., SVD of a tall-skinny matrix on a Hadoop cluster
- extending these algorithmic ideas to sparse matrices
  - e.g., sparse matrix-matrix multiplication
- using Strassen to do the rest of linear algebra in parallel
- trading off local memory and communication in parallel QR decomposition

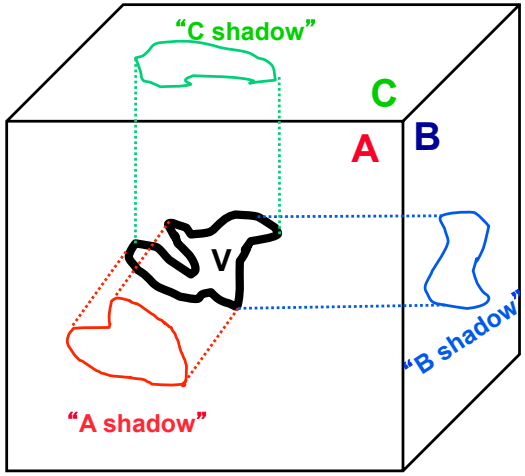
Main Idea of Lower Bound Proof

Crux of proof based on geometric inequality [Loomis & Whitney 49]



Volume of box

$$V = xyz = \sqrt{xz \cdot yz \cdot xy}$$



Volume of a 3D set

$$V \leq \sqrt{\text{area}(\text{A shadow})} \cdot \sqrt{\text{area}(\text{B shadow})} \cdot \sqrt{\text{area}(\text{C shadow})}$$

Given limited set of data, how much useful computation can be done?

Back

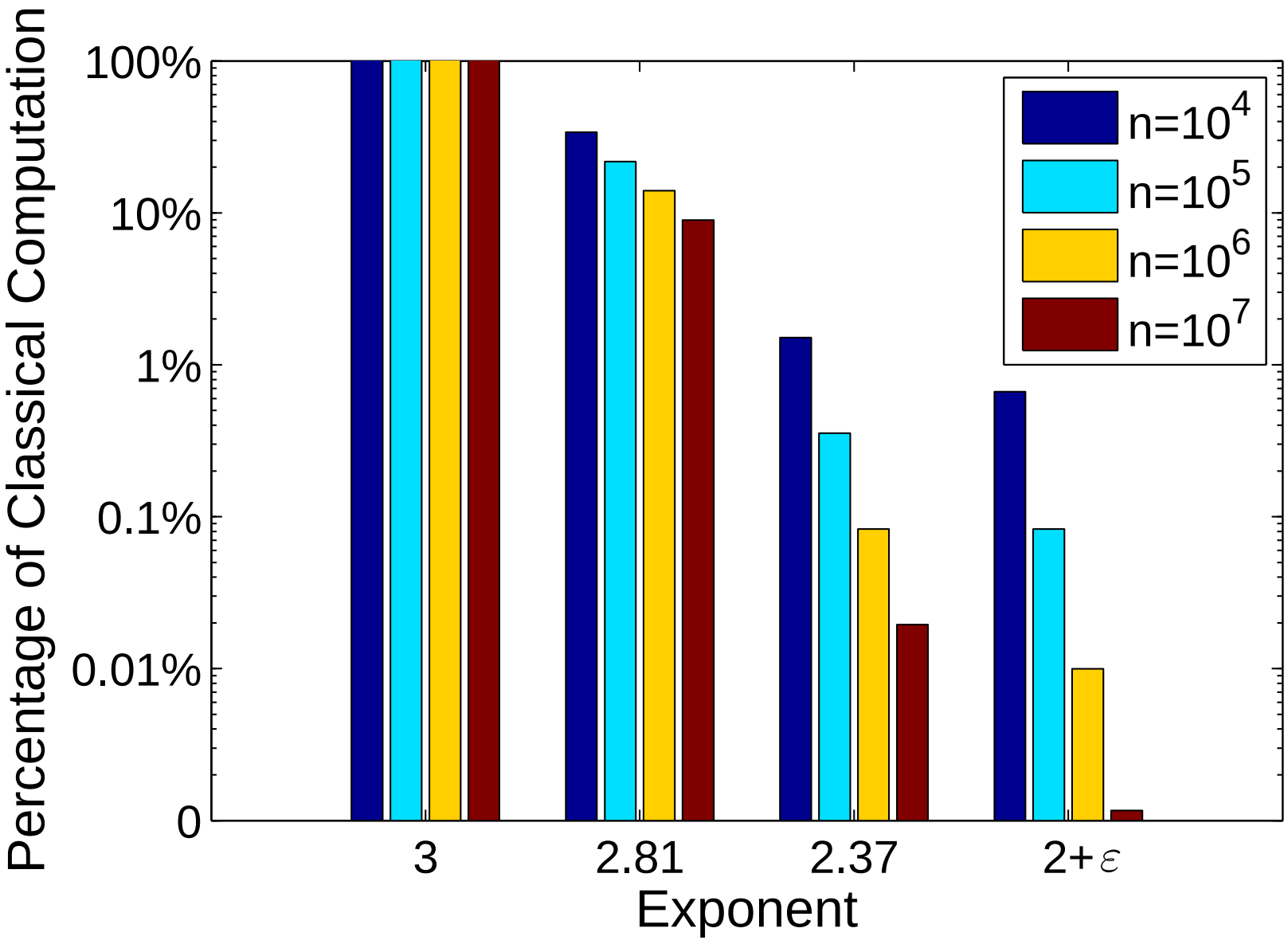
Can an $n \times n$ linear system of equations $Ax = b$ be solved in $O(n^{2+\varepsilon})$ operations, where ε is arbitrarily small?

. . . if solved affirmatively, [this] would change the world.

It is an article of faith for some of us that if $O(n^{2+\varepsilon})$ is ever achieved, the big idea that achieves it will correspond to an algorithm that is really practical.

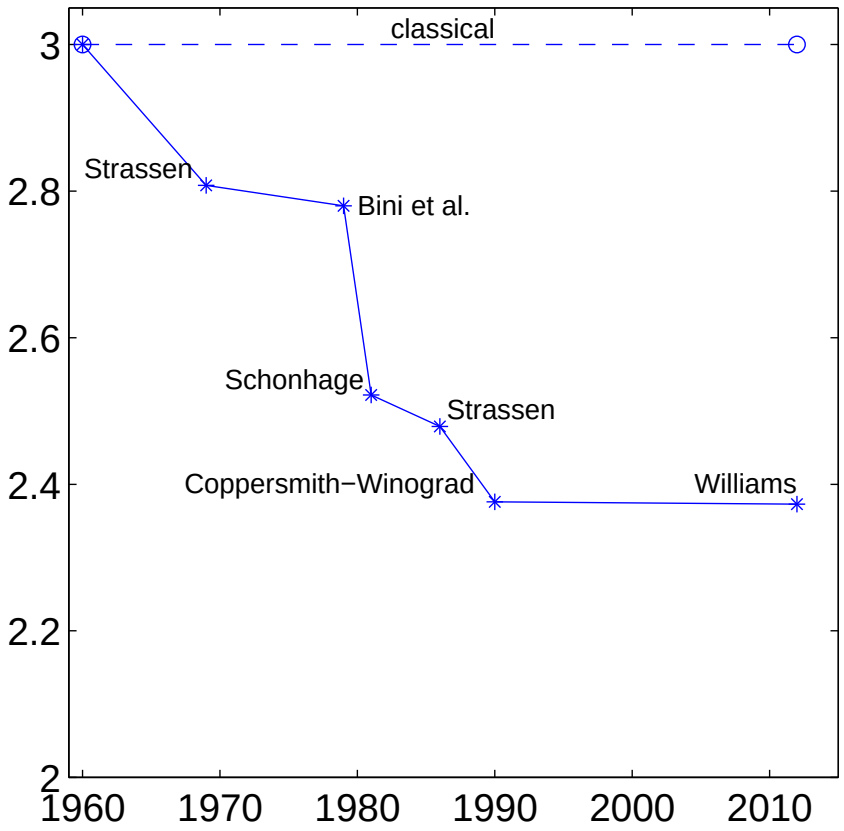
-Nick Trefethen, 2012 SIAM President

How much computation will that save?



Can we beat Strassen?

Exponent of matrix multiplication over time



Can we beat Strassen?

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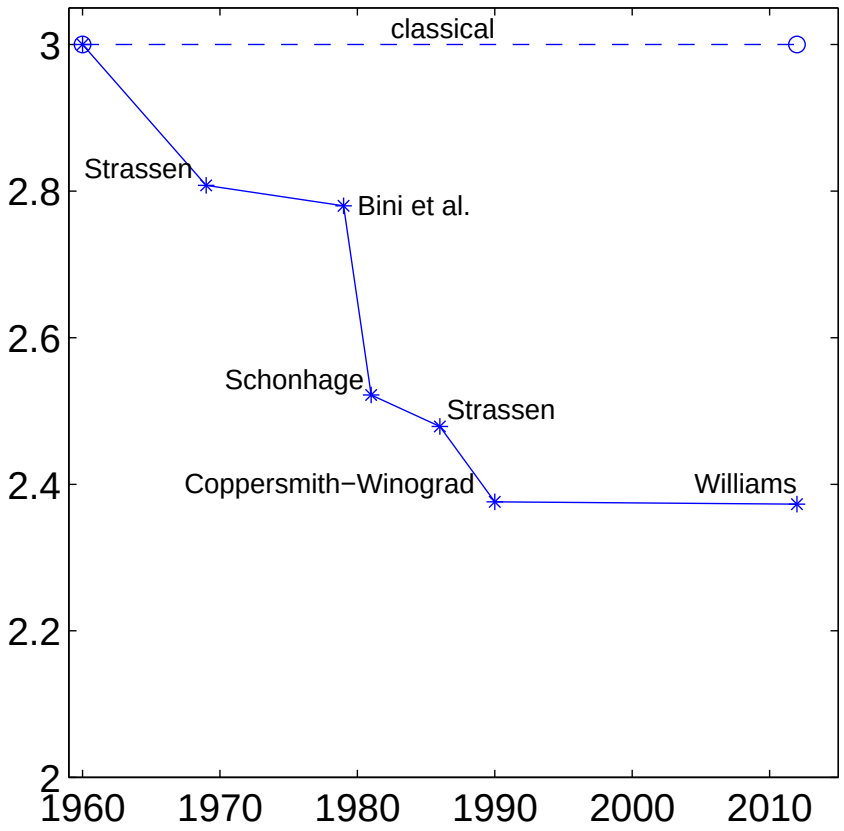

Unfortunately, these improvements are only theoretical because they

- involve approximations
- are existence proofs
- have (possibly) large constants

Solving the base case...

$2 \times 2 \times 2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

multiplies	6	7	8
flop count	$O\left(n^{2.58}\right)$	$O\left(n^{2.81}\right)$	$O\left(n^{3}\right)$

Solving the base case...

$2 \times 2 \times 2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

multiplies	~~6~~	7	8
flop count	~~$O\left(n^{2.58}\right)$~~	$O\left(n^{2.81}\right)$	$O\left(n^{3}\right)$

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multiplies	~~6~~	7	8
flop count	~~$O\left(n^{2.58}\right)$~~	$O\left(n^{2.81}\right)$	$O\left(n^{3}\right)$

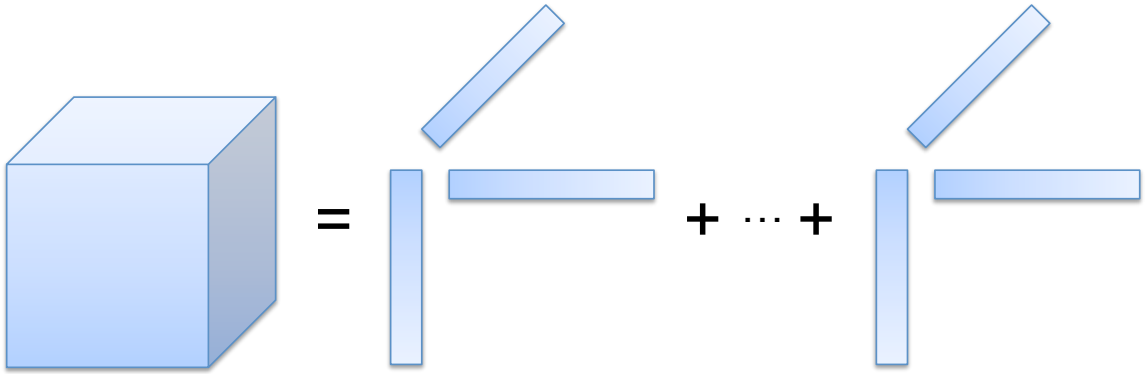
$3 \times 3 \times 3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

multiplies	19	21	23	27
flop count	$O\left(n^{2.68}\right)$	$O\left(n^{2.77}\right)$	$O\left(n^{2.85}\right)$	$O\left(n^{3}\right)$

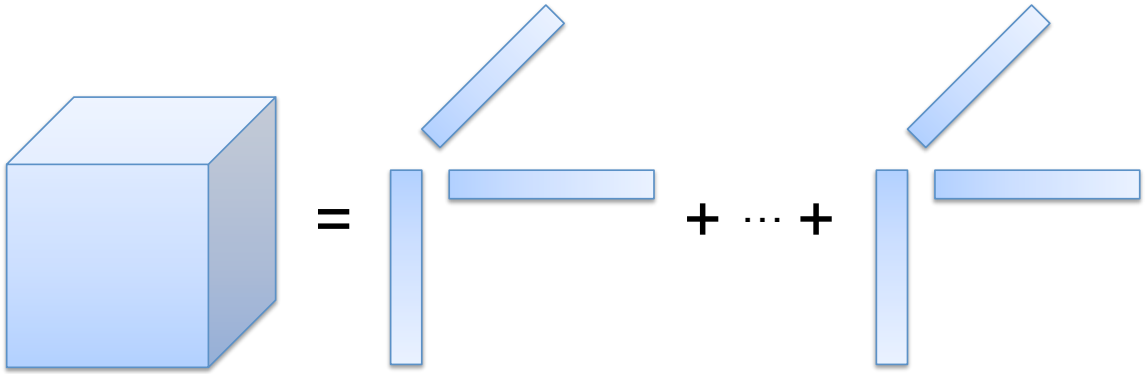
Beating Strassen

Finding a better base case corresponds to computing a low-rank decomposition of a particular 3D tensor



Beating Strassen

Finding a better base case corresponds to computing a low-rank decomposition of a particular 3D tensor



Unfortunately, this is a nonlinear integer optimization problem

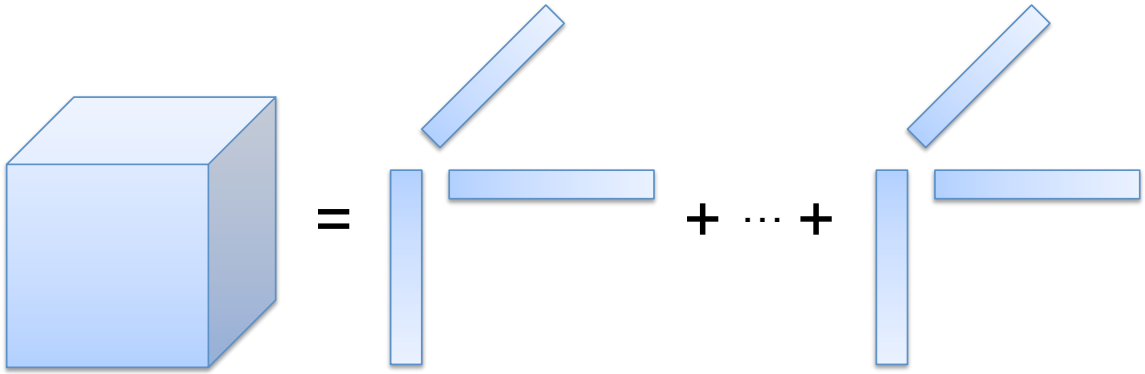
- it's NP-complete (in general), but need to solve it only once
- I used this method to re-discover Strassen

Could use (numerical) low-rank tensor approximation algorithms

- very efficient, but no guarantees

Beating Strassen

Finding a better base case corresponds to computing a low-rank decomposition of a particular 3D tensor



If we find it, we can make it practical!

- same parallelization as Strassen, but with less computation *and* communication

Memory-Independent Lower Bounds

	Classical	Strassen
Memory-dependent lower bound	$\Omega\left(\frac{n^3}{P\sqrt{M}}\right)$	$\Omega\left(\frac{n^\omega}{PM^{\omega/2-1}}\right)$
Memory-independent lower bound	$\Omega\left(\frac{n^2}{P^{2/3}}\right)$	$\Omega\left(\frac{n^2}{P^{2/\omega}}\right)$
Perfect strong scaling range	$P = O\left(\frac{n^3}{M^{3/2}}\right)$	$P = O\left(\frac{n^\omega}{M^{\omega/2}}\right)$
Attaining algorithm	[SD11]	[BDH+12b]

Example: Compute Eigenvalues of Band Matrix

Suppose we want to solve $Ax = \lambda x$ where A

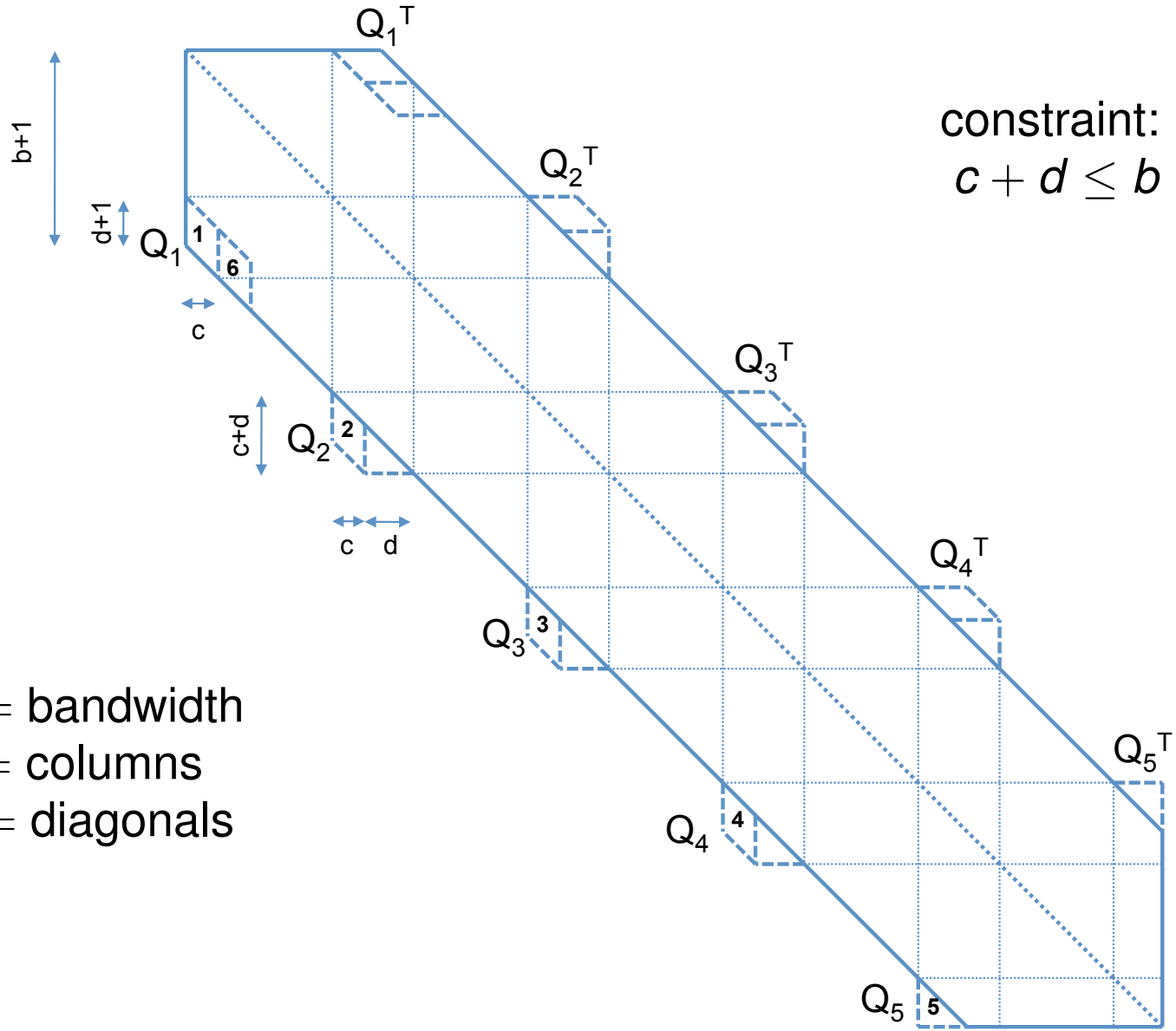
- is symmetric (save half the storage and flops)
- has band structure (exploit sparsity – ignore zeros)

We generally want to compute a factorization

$$A = QTQ^T$$

Q is an orthogonal matrix and T is symmetric tridiagonal

Successive Band Reduction (bulge-chasing)



constraint:
$c + d \leq b$

$b =$ bandwidth
$c =$ columns
$d =$ diagonals

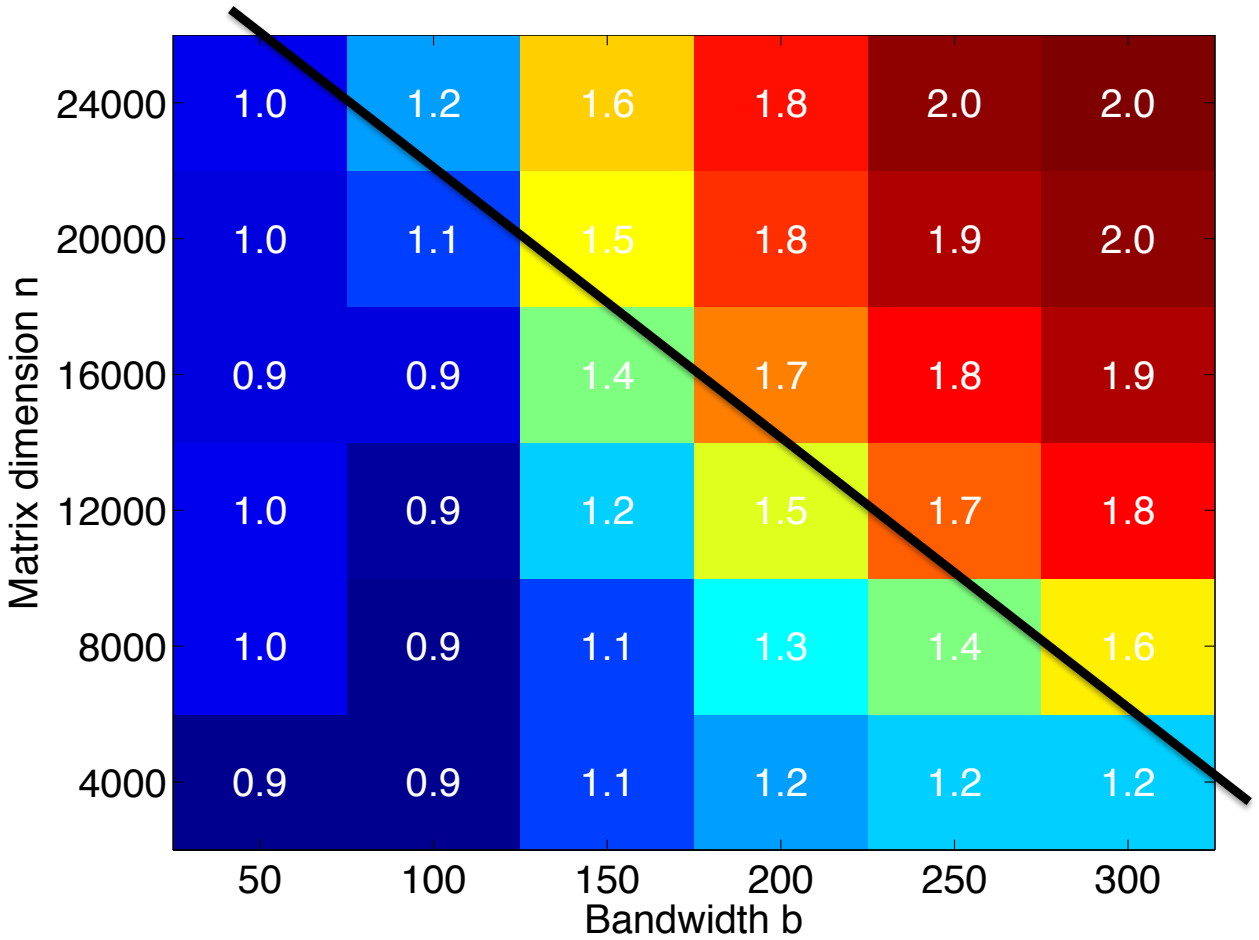
CASBR Data Access Pattern

One bulge at a time

Four bulges at a time

Implementation of Band Eigensolver (CASBR)

Speedup of sequential CASBR over Intel's Math Kernel Library

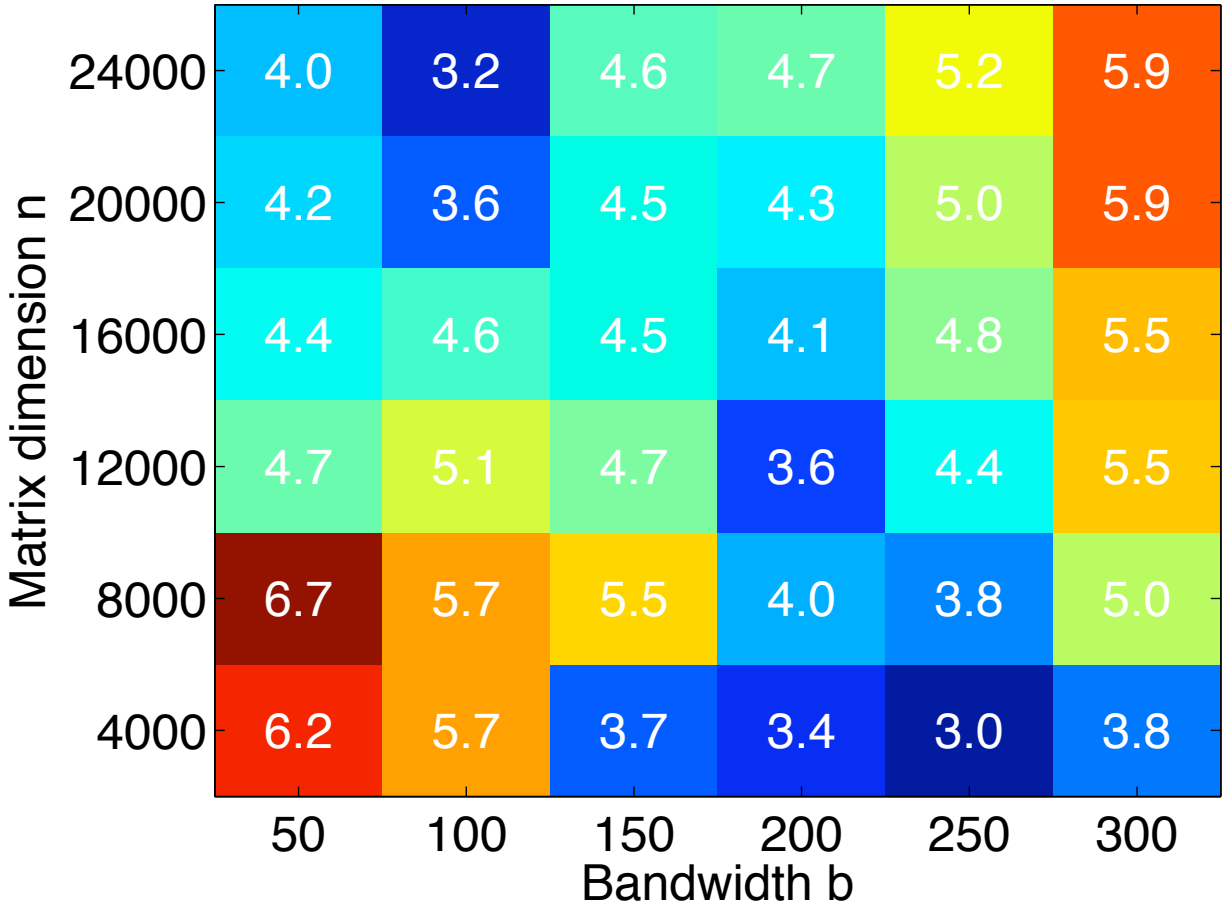


Benchmarked on Intel 10-core Westmere-EX socket [BDK12a]

▸ Algorithmic Details

Implementation of Band Eigensolver (CASBR)

Speedup of parallel CASBR (10 threads) over PLASMA library



Benchmarked on Intel 10-core Westmere-EX socket [BDK12a]

▸ Algorithmic Details

Example Application: Video Background Subtraction

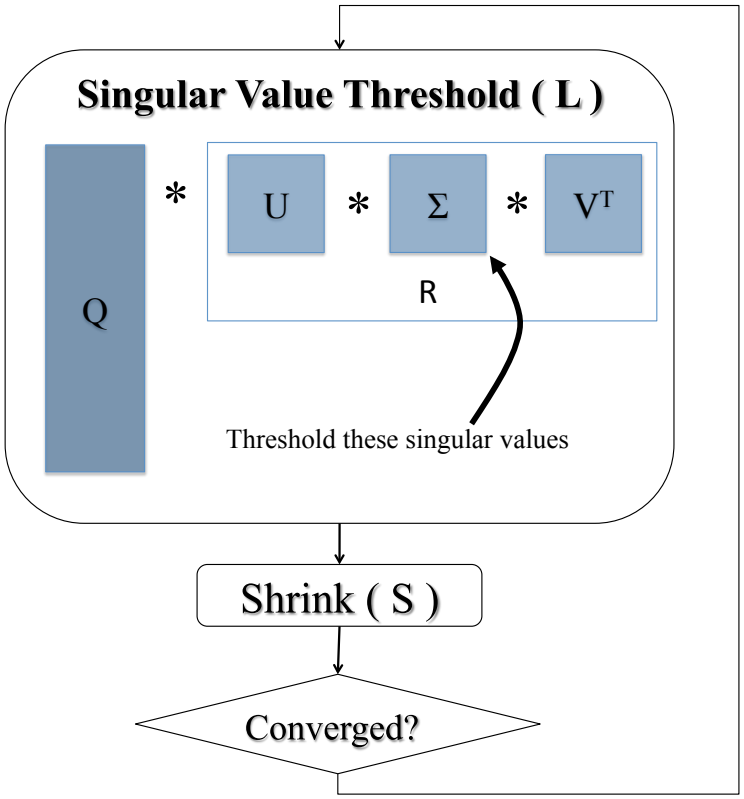
Idea: use Robust PCA algorithm [CLMW09] to subtract constant background from the action of a surveillance video

Given a matrix M whose columns represent frames, compute

$$M = L + S$$

where L is low-rank and S is sparse

Example Application: Video Background Subtraction



Compute:

$$M = L + S$$

where L is low-rank and S is sparse

The algorithm works iteratively, each iteration requires a singular value decomposition (SVD)

- M is $110{,}000 \times 100$

Communication-avoiding algorithm provided $3\times$ speedup over best GPU implementation [ABDK11]

Overview of Divide & Conquer Algorithm for Nonsymmetric Eigenproblem

One step of divide and conquer:

1. Compute $\left(I + (A^{-1})^{2^k}\right)^{-1}$ implicitly
   - maps eigenvalues of A to 0 and 1 (roughly)
2. Compute rank-revealing decomposition to find invariant subspace
3. Output block-triangular matrix

$$A_{\text{new}} = U^* A U = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon & A_{22} \end{bmatrix}$$

- block sizes chosen to minimize norm of ε
- eigenvalues of A_{11} all lie outside unit circle, eigenvalues of A_{22} lie inside unit circle, subproblems solved recursively
- stable, but progress guaranteed only with high probability

Reduction Example: LU

It's easy to reduce matrix multiplication to LU:

$$T \equiv \begin{bmatrix} I & 0 & -B \\ A & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & & \\ A & I & \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & -B \\ & I & A \cdot B \\ & & I \end{bmatrix} \equiv L \cdot U$$

- LU factorization can be used to perform matrix multiplication
- Communication lower bound for matrix multiplication applies to LU

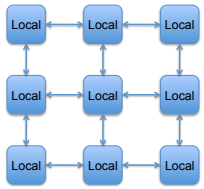
Reduction to Cholesky is a little trickier, but same idea [BDHS10]

Algorithms - Parallel $O(n^3)$ Linear Algebra

Algorithm	Reference	Factor exceeding lower bound for # words	Factor exceeding lower bound for # messages
Matrix Multiply	[Can69]	1	1
Cholesky	ScaLAPACK	$\log P$	$\log P$
Symmetric Indefinite	[BDD+12a] ScaLAPACK	proposed work $\log P$	proposed work $(N/P^{1/2}) \log P$
LU	[GDX11] ScaLAPACK	$\log P$ $\log P$	$\log P$ $(N/P^{1/2}) \log P$
QR	[DGHL12] ScaLAPACK	$\log P$ $\log P$	$\log^3 P$ $(N/P^{1/2}) \log P$
SymEig, SVD	[BDK12a] ScaLAPACK	proposed work $\log P$	proposed work $N/P^{1/2}$
NonsymEig	[BDD12b] ScaLAPACK	$\log P$ $P^{1/2} \log P$	$\log^3 P$ $N \log P$

*This table assumes that *one* copy of the data is distributed evenly across processors

Red = not optimal



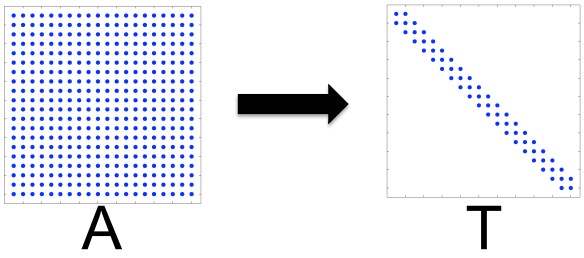
Back

Symmetric Eigenproblem and SVD via SBR

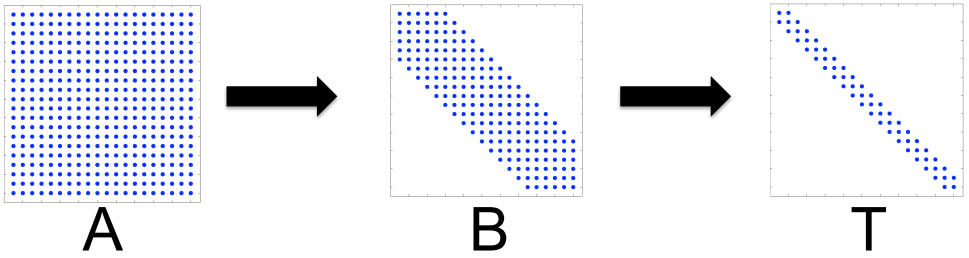
We're solving the symmetric eigenproblem via reduction to tridiagonal form

- Conventional approach (e.g. LAPACK) is direct tridiagonalization
- Two-phase approach reduces first to band, then band to tridiagonal

Direct:



Two-step:



- first phase can be done efficiently
- second phase is trickier, requires successive band reduction (SBR) [BLS00]
  - involves "bulge-chasing"
  - we've improved it to reduce communication [BDK12b]

Back

Communication-Avoiding SBR - theory

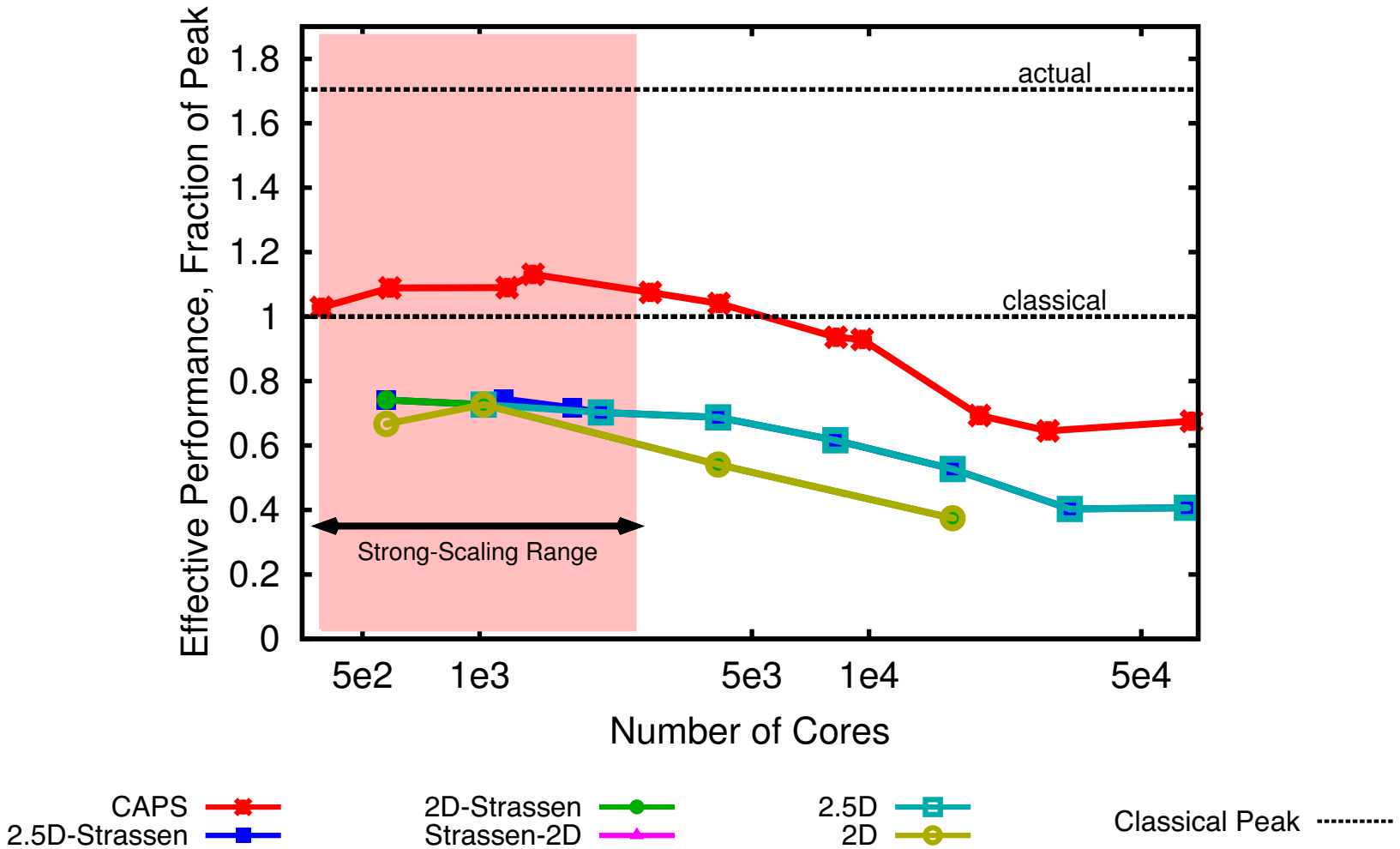
	Flops	Words Moved	Data Re-use
Schwarz	$4n^2b$	$O(n^2b)$	$O(1)$
M-H	$6n^2b$	$O(n^2b)$	$O(1)$
B-L-S*	$5n^2b$	$O(n^2 \log b)$	$O\left(\frac{b}{\log b}\right)$
CA-SBR†	$5n^2b$	$O\left(\frac{n^2b^2}{M}\right)$	$O\left(\frac{M}{b}\right)$

*with optimal parameter choices
†assuming $1 \leq b \leq \sqrt{M}/3$

Back

Performance of CAPS on large problems

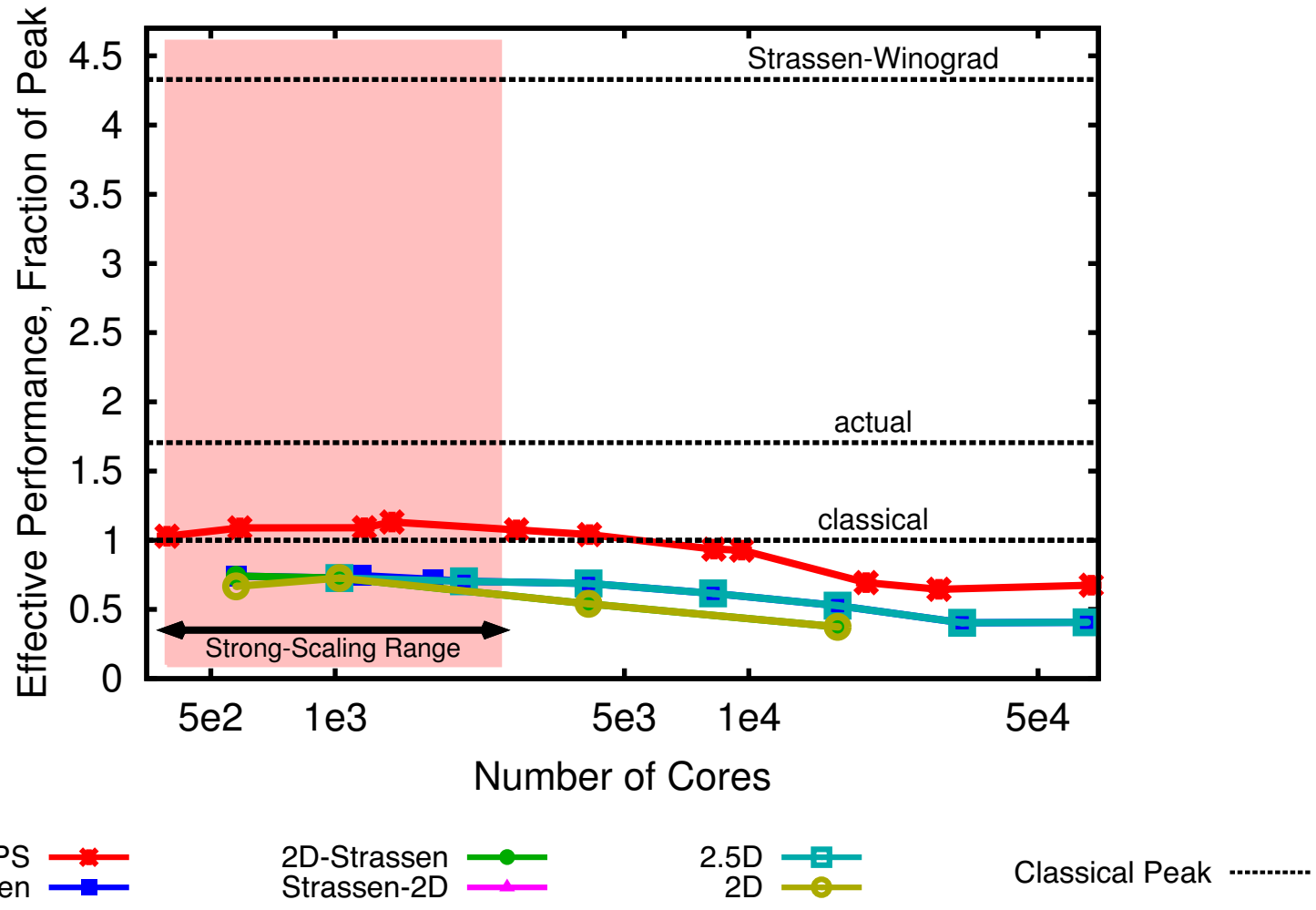
Strong-scaling on Intrepid (IBM BG/P), $n = 65{,}856$.



Back

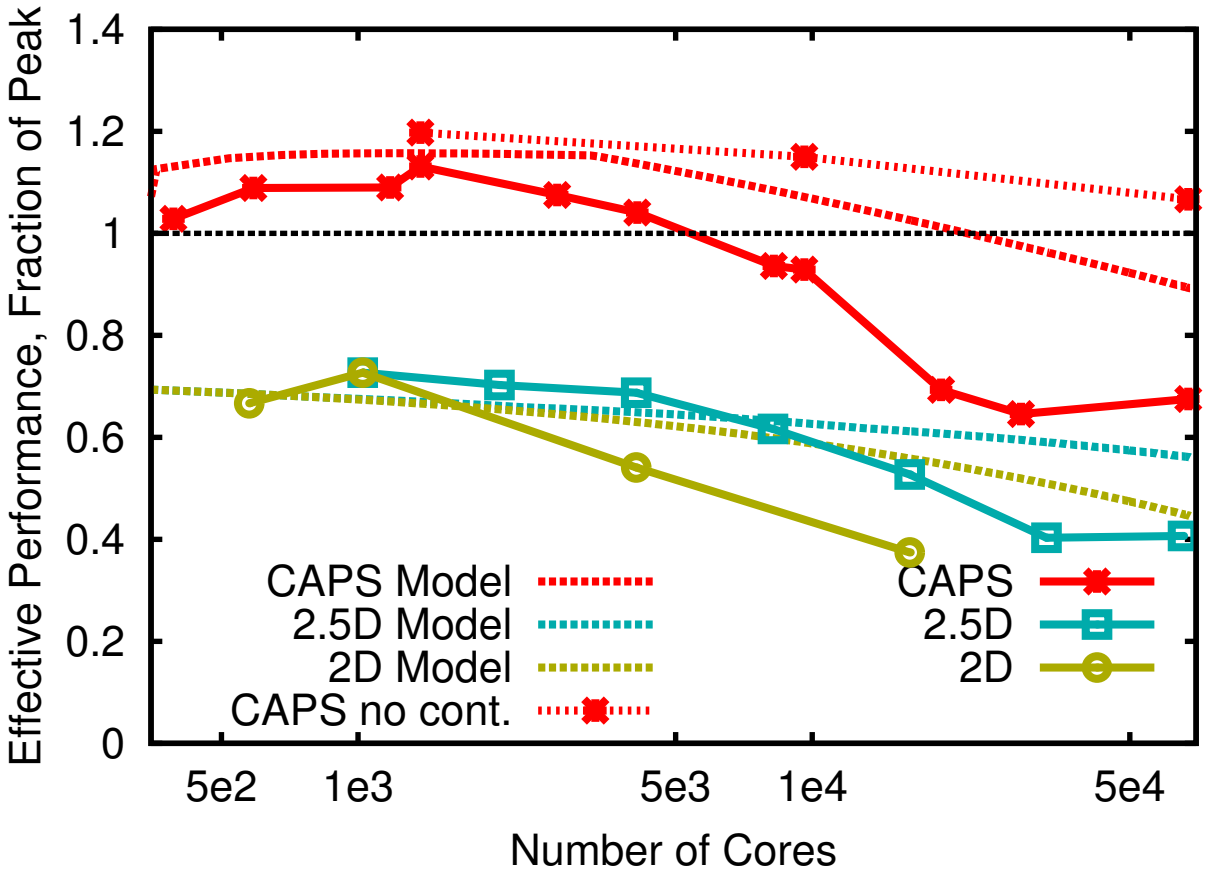
Performance of CAPS on large problems

Strong-scaling on Intrepid (IBM BG/P), $n = 65{,}856$.



Back

Performance: Model vs Actual



Comparison of the parallel models with the algorithms in strong scaling of matrix dimension $n = 65{,}856$ on Intrepid.

Back

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Avoiding Communication in Dense Linear Algebra

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Thank You!

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